MATH2

1) $h(x,y,z,\lambda) = ax+by+cz-\lambda(x^2+y^2+z^2)$

$$\left.\begin{aligned} h_x=0 &\Rightarrow a-2\lambda x=0\\ h_y=0 &\Rightarrow b-2\lambda y=0\\ h_z=0 &\Rightarrow c-2\lambda z=0\end{aligned}\right\} \quad x^2+y^2+z^2=R^2 \Rightarrow \frac{1}{4\lambda^2}(a^2+b^2+c^2)=R^2$$

at min/max $\qquad \Rightarrow \lambda = \pm\sqrt{a^2+b^2+c^2}/2R$

$ax+by+cz = \frac{1}{2\lambda}(a^2+b^2+c^2) = \pm R\sqrt{a^2+b^2+c^2}$ (as expected geometrically) (dist of plane from O, tangent to sphere)

(+ sign gives max
− sign gives min)

2) $f(x,y)=e^{xy}$, $f_x = ye^{xy}$, $f_y = xe^{xy}$, $f_{xx}=y^2e^{xy}$, $f_{yy}=x^2e^{xy}$

$f_{xy} = e^{xy}+xye^{xy}$.

$f(x_0+h, y_0+k) = f(x_0,y_0) + h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y} + \frac{1}{2}\left(h^2\frac{\partial^2 f}{\partial x^2} + 2hk\frac{\partial^2 f}{\partial x\partial y} + k^2\frac{\partial^2 f}{\partial y^2}\right)+\cdots$

& evaluating at $x_0=y_0=1$, $h=(x-1)$, $k=y-1$ $\Rightarrow$

$f(x,y) = e + (x-1)e + (y-1)e + \frac{1}{2}\left((x-1)^2e + 2(x-1)(y-1).2e + (y-1)^2.e\right)+\cdots$

$= e\left(1+(x-1)+(y-1)+\frac{1}{2}(x-1)^2 + 2(x-1)(y-1) + \frac{1}{2}(y-1)^2\right)$

3) $\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$

Consider $\frac{d}{dx}\left(F - y'\frac{\partial F}{\partial y'}\right) = \underbrace{\frac{\partial F}{\partial x}}_{=0} + y'\frac{\partial F}{\partial y} + \cancel{y''\frac{\partial F}{\partial y'}} - \cancel{y''\frac{\partial F}{\partial y'}} - y'\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)$

$= y'\left(\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right) = y'.0 = 0$

$\Rightarrow F - y'\frac{\partial F}{\partial y'} = \text{Constant}$

4) $xu_x + y^2u_y = u$. χ eqns are $dx = x\,dt$, $dy = y^2dt$, $du = u\,dt$

So. $d(x/u) = \frac{dx}{u} - \frac{1}{u^2}x\,du = \left(\frac{x}{u}-\frac{x}{u}\right)dt = 0$, $d(\ln x + 1/y) =$

$\frac{1}{x}dx - \frac{1}{y^2}dy = dt - dt = 0 \Rightarrow x/u$ & $\ln x + 1/y$ are constants. Lagrange method says soln is $x/u = F(\ln x + 1/y)$. $u(1,y)=y \Rightarrow 1/y = F(1/y)$

$\Rightarrow F(s)=s$ & $u = \frac{x}{\ln x + 1/y} = \frac{xy}{1+y\ln x}$

5) $z_{xx} = 1/c^2 \, z_{tt}$. If $z = f(x+mt)$ then $f'' = c^2/m^2 f''$

General solution is thus $z = f(x-ct) + g(x+ct)$

Initial conditions $\Rightarrow$ $z(0,t) = F(x) = f(x) + g(x)$

$z_t(0,t) = G(x) = -cf'(x) + cg'(x)$

$\Rightarrow -f + g = \frac{1}{c}\int_{\alpha}^{x} G(\xi)\,d\xi$ so $g = \frac{1}{2}F + \frac{1}{2c}\int_{\alpha}^{x} G(\xi)\,d\xi$

$f = +\frac{1}{2}F - \frac{1}{2c}\int_{\alpha}^{x} G(\xi)\,d\xi$

so $z(x,t) = \frac{1}{2}\left(F(x-ct) + F(x+ct)\right) + \frac{1}{2c}\left\{\int_{x-ct}^{\alpha} G(\xi)\,d\xi + \int_{\alpha}^{x+ct} G(\xi)\,d\xi\right.$

$= \frac{1}{2}\left(F(x-ct) + F(x+ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} G(\xi)\,d\xi$.

6) $u_{tt} + ku_t = c^2 u_{xx} + bu$. If $u = X(x)T(t)$, then

$T''X + kT'X = c^2 X'' T + bXT \Rightarrow \frac{T''}{T} + \frac{kT'}{T} = c^2\frac{X''}{X} + b$

l.h.s is indep of x, r.h.s of t so both must be constant, λ say $\Rightarrow$ $T'' + kT' - \lambda T = 0$ & $c^2X'' + (b-\lambda)X = 0$.

7) a) Form $\int_0^1 y'^2 - \lambda y^2 \, dy$ & EL eqns give, $\frac{\partial F}{\partial y} - \frac{d}{dx}$

$$-2\lambda y - \frac{d}{dx}(2y') = 0 \Rightarrow y'' + \lambda y = 0$$

BCs are $y(0) = y(1) = 0$. These can only be satisfied by choosing $\lambda > 0$, $= p^2$, $y = A\sin px + B\cos px$. BC's give $B = 0,\ p = n\pi$

$y = A\sin n\pi x$. Constraint $\int_0^1 y^2 dx = 1$ gives $A^2 \cdot \frac{1}{2} = 1 \Rightarrow A = \sqrt{2}$

$$\underline{y(x) = \sqrt{2}\sin(n\pi x)}$$

(examples of eigenvalue problems not seen, isoperimetric problems seen)

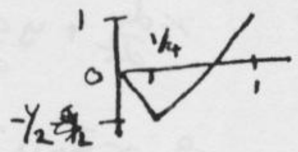
b) $I[y] = \int_0^1 y'^2 f(x)\, dx$. EL eqns give $0 - \frac{d}{dx}(2y'f) = 0$

$\Rightarrow y'f = \text{const} = c$

For $x < 1/4$ $y' = -c$ & with $y(0) = 0$, $y = -cx$, $y(1/4) = -c/4$

For $x > 1/4$ $y' = c$ & with $y(1/4) = -c/4$, $y = -\frac{c}{4} + (x - 1/4)c$

$y(1) = 1 \Rightarrow 1 = -\frac{c}{4} + \frac{3}{4}c = c/2 \Rightarrow c = 2$



Extremal curve is $y(x) = -2x \quad 0 \le x \le 1/4$

$y(x) = 2x - 1 \quad 1/4 \le x \le 1$

$$I = \int_0^{1/4} 4\cdot(-1)\,dx + \int_{1/4}^1 4(1)\,dx = -1 + 3 = \underline{2}$$

[Nothing of this type seen, other than observation $\frac{\partial F}{\partial y} = 0 \Rightarrow \frac{\partial F}{\partial y'} = \text{const}$

No discussion of piecewise constant $y'f$ expected]

8 a) $x u_x + y u_y = x(x^2+y^2)u$

$\frac{dx}{x} = \frac{dy}{y} = \frac{du}{ux(x^2+y^2)}$ $\Rightarrow$ $\frac{dy}{dx} = \frac{y}{x}$, $\frac{dy}{y} = \frac{dx}{x}$ $\Rightarrow$ $y = \varphi x$.

y/x constant $= \varphi$. Change variables to φ & x:

$$x\left(u_x + u_\varphi\left(-\frac{y}{x^2}\right)\right) + y\left(u_\varphi\left(\frac{1}{x}\right)\right) = x(x^2+x^2\varphi^2)u.$$

$$\Rightarrow \cancel{\frac{d}{dx}}\ \frac{\partial u}{\partial x} = x^2(1+\varphi^2)u \Rightarrow u = e^{\frac{1}{3}x^3(1+\varphi^2)} f(\varphi)$$

$$= f(y/x)\, e^{1/3(x^3+xy^2)}$$

b) $(3y-2u)u_x + (u-3x)u_y = (2x-y)$

$\frac{dx}{dt} = 3y-2u$, $\frac{dy}{dt} = u-3x$, $\frac{du}{dt} = 2x-y$

So, we observe: $\frac{dx}{dt} + 2\frac{dy}{dt} + 3\frac{dy}{dt} = 3y-2u+2u-6x+6x-3y = 0$

$\Rightarrow x+2y+3u = C$

Also $x\frac{dx}{dt} + y\frac{dy}{dt} + u\frac{du}{dt} = 3xy-2xu+uy-3xy+2xu-yu = 0$

$\Rightarrow x^2+y^2+u^2 = C$

So $x^2+y^2+u^2 = f(x+2y+3u)$

$u(x,x) = 0 \Rightarrow 2x^2 = f(3x) \Rightarrow x^2+y^2+u^2 = \frac{2(x+2y+3u)^2}{9}$.

c) $u_t + u^2u_x = 0 \Rightarrow u = $ const on lines

$\frac{dx}{dt} = u^2$ & $x = u^2t + C$, $u(x,0) = x \Rightarrow C = u$

& $x = u^2t+u$, $u^2t+u-x = 0$ $u = \frac{-1 \pm \sqrt{1+4xt}}{2t}$

$+$ sign allows $u(x,0)$ finite & $u = \frac{\sqrt{1+4xt}-1}{2t}$

9) $T_{xx} = T_t + f(x)$ $\quad T(0) = 0, \quad T(1) = Q.$

$T = S + \theta \Rightarrow S'' + \theta_{xx} = \theta_t + f$. Choose $\theta_t = \theta_{xx}$ & $S'' = f$.

Solve $S'' = f$ with b.c. $S(0) = 0$, $S(1) = Q$ to give

$S' = P + \int_0^x f(q)\,dq \Rightarrow S = Px + \int_0^x \int_0^q f(\tilde{q})\,d\tilde{q}\,dq$ using $S(0) = 0$

$S(1) = Q \Rightarrow Q = P + \int_0^1 \int_0^q f(\tilde{q})\,d\tilde{q}\,dq.$

$\theta_{xx} = \theta_t$ & with $\theta(0) = 0$, $\theta(1) = 0$, standard separation of variables gives $\theta(x,t) = \sum_{n=1}^{\infty} e^{-n^2\pi^2 t} \sin(n\pi x) A_n.$

If $f(x) = 1$ then, going back to $S'' = f = 1$, $S' = A + x$, $Q = 0$.

$S = Ax + \frac{1}{2}x^2$ using $S(0) = 0$ & $S(1) = 0 \Rightarrow A = -1/2$. $S = \frac{1}{2}(x^2 - 1)$

Initially $T = 0$, so $0 = T = \theta(x,0) + S$. So $\theta(x,0) = -S$

So $\sum_1^{\infty} \sin(n\pi x) A_n = \frac{1}{2}(1 - x^2)$ & $T(x,t) = \frac{1}{2}(x^2 - 1) + \sum_1^{\infty} e^{-n^2\pi^2 t} A_n \sin n\pi x$

$\Rightarrow A_n \cdot \frac{1}{2} = \int_0^1 \frac{1}{2}(1 - x^2) \sin n\pi x\,dx$

$$A_n = \left\{ \left[(1 - x^2) \frac{\cos n\pi x}{-n\pi} \right]_0^1 + \int_0^1 \frac{\cos n\pi x}{n\pi}(-2x)\,dx \right\} = \left\{ \frac{1}{n\pi} + \left[-2x \frac{\sin n\pi x}{(n\pi)^2} \right]_0^1 + \int_0^1 2 \frac{\sin n\pi x}{(n\pi)^2}\,dx \right\} = \left\{ \frac{1}{n\pi} + 0 + \left[2 \frac{\cos n\pi x}{n^3\pi^3} (-1) \right]_0^1 \right\}$$

$$= \frac{1}{n\pi} + \frac{2}{n^3\pi^3}\left(1 - (-1)^n\right).$$